Rotations

Learning Targets: 8, 17, 20, 21, 23, 24,

Learning Objectives

• Find the image of a figure in a rotation in a coordinate plane.

• Recognize that a rotation is an isometry.

Review Queue

a. Reflect $\triangle XYZ$ with vertices $X(9, 2), Y(2, 4)$ and $Z(7, 8)$ over the $y-$axis. What are the vertices of $\triangle X'Y'Z'$?

b. Reflect $\triangle XYZ$ over the $x-$axis. What are the vertices of $\triangle X''Y''Z''$?

c. How do the coordinates of $\triangle X''Y''Z''$ relate to $\triangle XYZ$?

Know What? The international symbol for recycling appears below. It is three arrows rotated around a point. Let’s assume that the arrow on the top is the preimage and the other two are its images. Find the center of rotation and the angle of rotation for each image.

Defining Rotations

**Rotation:** A transformation by which a figure is turned around a fixed point to create an image.

**Center of Rotation:** The fixed point that a figure is rotated around.

Lines can be drawn from the preimage to the center of rotation, and from the center of rotation to the image. The angle formed by these lines is the angle of rotation.

In this section, our center of rotation will always be the origin. Rotations can also be clockwise or counterclockwise. We will only do **counterclockwise** rotations, to go along with the way the quadrants are numbered.
**Investigation:** Drawing a Rotation of $100^\circ$

Tools Needed: pencil, paper, protractor, ruler

a. Draw $\triangle ABC$ and a point $R$ outside the circle.

b. Draw the line segment $RB$.

c. Take your protractor, place the center on $R$ and the initial side on $RB$. Mark a $100^\circ$ angle.

d. Find $B'$ such that $RB = RB'$.

e. Repeat steps 2-4 with points $A'$ and $C'$.

f. Connect $A'$, $B'$, and $C'$ to form $\triangle A'B'C''$. 
This is the process you would follow to rotate any figure \( 100^\circ \) counterclockwise. If it was a different angle measure, then in Step 3, you would mark a different angle. You will need to repeat steps 2-4 for every vertex of the shape.

**180° Rotation**

To rotate a figure \( 180^\circ \) in the coordinate plane, we use the origin as the center of the rotation. Recall, that a \( 180^\circ \) angle is the same as a straight line. So, a rotation of a point over the origin of \( 180^\circ \) will be on the same line and the same distance away from the origin.

**Example 1:** Rotate \( \triangle ABC \), with vertices \( A(7,4), B(6,1), \) and \( C(3,1) \) \( 180^\circ \). Find the coordinates of \( \triangle A'B'C' \).

**Solution:** You can either use Investigation 12-1 or the hint given above to find \( \triangle A'B'C' \). It is very helpful to graph the triangle. Using the hint, if \( A \) is \((7,4)\), that means it is 7 units to the right of the origin and 4 units up. \( A' \) would then be 7 units to the left of the origin and 4 units down. The vertices are:

- \( A(7,4) \rightarrow A'(-7,-4) \)
- \( B(6,1) \rightarrow B'(-5,-1) \)
- \( C(3,1) \rightarrow C'(-3,-1) \)

The image has vertices that are the negative of the preimage. This will happen every time a figure is rotated \( 180^\circ \).

**Rotation of \( 180^\circ \):** If \((x,y)\) is rotated \( 180^\circ \) around the origin, then the image will be \((-x,-y)\).

From this example, we can also see that a rotation is an isometry. This means that \( \triangle ABC \cong \triangle A'B'C' \). You can use the distance formula to verify that our assertion holds true.

**90° Rotation**

Similar to the \( 180^\circ \) rotation, a \( 90^\circ \) rotation (counterclockwise) is an isometry. Each image will be the same distance away from the origin as its preimage, but rotated \( 90^\circ \).

**Example 2:** Rotate \( ST \) \( 90^\circ \).
Solution: When we rotate something $90^\circ$, you can use Investigation 12-1. Draw lines from the origin to $S$ and $T$. The line from each point to the origin is going to be perpendicular to the line from the origin to its image. Therefore, if $S$ is 6 units to the right of the origin and 1 unit down, $S'$ will be 6 units up and 1 to the right.

Using this pattern, $T'$ is (8, 2).

If you were to write the slope of each point to the origin, $S$ would be $\frac{-1}{6} \rightarrow \frac{1}{2}$, and $S'$ must be $\frac{1}{6} \rightarrow \frac{-1}{2}$. Again, they are perpendicular slopes, following along with the $90^\circ$ rotation. Therefore, the $x$ and the $y$ values switch and the new $x$ value is the opposite sign of the original $y$ value.

Rotation of $90^\circ$: If $(x, y)$ is rotated $90^\circ$ around the origin, then the image will be $(-y, x)$.

Rotation of $270^\circ$

A rotation of $270^\circ$ counterclockwise would be the same as a clockwise rotation of $90^\circ$. We also know that a $90^\circ$ rotation and a $270^\circ$ rotation are $180^\circ$ apart. We know that for every $180^\circ$ rotation, the $x$ and $y$ values are negated. So, if the values of a $90^\circ$ rotation are $(-y, x)$, then a $270^\circ$ rotation would be the opposite sign of each, or $(y, -x)$.

Rotation of $270^\circ$: If $(x, y)$ is rotated $270^\circ$ around the origin, then the image will be $(y, -x)$.

Example 3: Find the coordinates of $ABCD$ after a $270^\circ$ rotation.
Solution: Using the rule, we have:

While we can rotate any image any amount of degrees, only $90^\circ$, $180^\circ$, and $270^\circ$ have special rules. To rotate a figure by an angle measure other than these three, you must use Investigation 12-1.

Example 4: Algebra Connection The rotation of a quadrilateral is shown below. What is the measure of $x$ and $y$?

Solution: Because a rotation is an isometry, we can set up two equations to solve for $x$ and $y$.

\[
\begin{align*}
2y &= 80^\circ \\
y &= 40^\circ \\
2x - 3 &= 15 \\
x &= 9
\end{align*}
\]

Know What? Revisited The center of rotation is shown in the picture below. If we draw rays to the same point in each arrow, we see that the two images are a $120^\circ$ rotation in either direction.
Degree (Order) of Rotational Symmetry of a Regular Polygon

The degree or order of rotational symmetry of a regular (remember! Regular means that all the sides are the same length and all the angles are the same size) polygon is the number of rotations a shape will take to rotate back to the beginning.

For example:

A regular hexagon has 6 lines of symmetry. Notice the center of rotation is where the symmetry lines intersect at the center of the figure. Also notice there are twice as many angles as sides of the polygon. Since these angles go around in a circle in the center of the shape, the degree of each of these central angles are 360 / 12. Thus each angle measures 30 degrees. The number of 30 degree turns the upper right vertex would need to rotate to get back to where it started = 360 / 30 = 12.

In general, the degree of rotational symmetry of a regular polygon = 360 / 2*the number of sides.

Review Questions

In the questions below, every rotation is counterclockwise, unless otherwise stated.

Using patty paper, trace the shape and rotate each figure around point $P$ the given angle measure.

1. $50^\circ$

2. $120^\circ$

3. $200^\circ$
Rotate each figure in the coordinate plane the given angle measure. The center of rotation is the origin.

4. $90^\circ$

5. $270^\circ$

6. $270^\circ$

7. $90^\circ$

**Algebra Connection** Find the measure of $\theta$ in the rotations below. The blue figure is the preimage.
Find the angle of rotation for the graphs below. The center of rotation is the origin and the blue figure is the preimage.
14. Use what you did in #’s 11-13 to define rotation.

Two Reflections The vertices of $\triangle GHI$ are $G(-2, 2)$, $H(8, 2)$, and $I(6, 8)$. Use this information to answer questions 24-27.

15. Plot $\triangle GHI$ on the coordinate plane.

16. Reflect $\triangle GHI$ over the $x-$axis. Find the coordinates of $\triangle G'H'I'$.

17. Reflect $\triangle G'H'I'$ over the $y-$axis. Find the coordinates of $\triangle G''H''I''$.

18. What one transformation would be the same as this double reflection?

Multistep Construction Problem

19. Draw two lines that intersect, $m$ and $n$, and $\triangle ABC$. Reflect $\triangle ABC$ over line $m$ to make $\triangle A'B'C'$. Reflect $\triangle A'B'C'$ over line $n$ to get $\triangle A''B''C''$. Make sure $\triangle ABC$ does not intersect either line.

20. Draw segments from the intersection point of lines $m$ and $n$ to $A$ and $A''$. Measure the angle between these segments. This is the angle of rotation between $\triangle ABC$ and $\triangle A''B''C''$.

21. Measure the angle between lines $m$ and $n$. Make sure it is the angle which contains $\triangle A'B'C''$ in the interior of the angle.

22. What is the relationship between the angle of rotation and the angle between the two lines of reflection?

23. What is the degree of a regular octagon?

Review Queue Answers

a. $X'(-9, 2), Y'(-2, 4), Z'(-7, 8)$

b. $X''(-9, -2), Y''(-2, -4), Z''(-7, -8)$

c. $\triangle X''Y''Z''$ is the double negative of $\triangle XYZ: (x, y) \rightarrow (-x, -y)$